Introduction to Span

Definition: We call the set of all linear combinations of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in $\mathbb{R}^m$ the span of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ which we denote $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$.

We say that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ span (or generate) $\mathbb{R}^m$ if $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) = \mathbb{R}^m$.

Example 1: Is the vector $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \tag{1}$$

Explain.

$$\underbrace{x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3}_{} = \vec{b}\ ? \quad \times$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 1 & -1 & -1 \\ -4 & 3 & 0 \end{bmatrix}\vec{x} = \vec{b}\ ? \quad \times$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 1 & -1 & -1 & 0 \\ -4 & 3 & 0 & 0 \end{array}\right] \begin{array}{l} R_3 := R_3 + 4R_1 \\ R_2 := R_2 - R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & -1 & 4 & -1 \\ 0 & 3 & 12 & 4 \end{array}\right]$$

$$R_3 := R_3 + 3R_2 \left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & -1 & -4 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \begin{array}{r} x_1 + 3x_3 = 1 \\ -x_2 - 4x_3 = 1 \\ 0 = 1 \ \lightning \end{array}$$

Inconsistent

$\vec{b}$ is not in $\text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$

Theorem 1 (Poole 2.4): Let A be a $m \times n$ matrix with column vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and $\mathbf{b}$ be a vector in $\mathbb{R}^m$. The linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only of $\mathbf{b}$ is in $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$.

$$\vec{b} \text{ in } \text{span}(\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n) \iff x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{b}$$

$$\iff \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}\vec{x} = \vec{b} \iff A\vec{x} = \vec{b}.$$